

ON THE DETOUR AND VERTEX DETOUR HULL NUMBERS OF A GRAPH

A.P. SANTHAKUMARAN¹ AND S.V. ULLAS CHANDRAN²

¹Department of Mathematics Hindustan University Hindustan Institute of
Technology and Science, Padur, Chennai - 603 103, India

apskumar1953@yahoo.co.in

²Department of Mathematics, Mahatma Gandhi College, Kesavdasapuram,
Pattom P.O., Thiruvananthapuram - 695 004, India

svuc.math@gmail.com

Abstract. For vertices x and y in a connected graph G , the detour distance $D(x, y)$ is the length of a longest $x - y$ path in G . An $x - y$ path of length $D(x, y)$ is an $x - y$ detour. The closed detour interval $I_D[x, y]$ consists of x, y , and all vertices lying on some $x - y$ detour of G ; while for $S \subseteq V(G)$, $I_D[S] = \bigcup_{x, y \in S} I_D[x, y]$. A set S of vertices is a detour convex set if $I_D[S] = S$. The detour convex hull $[S]_D$ is the smallest detour convex set containing S . The detour hull number $dh(G)$ is the minimum cardinality among subsets S of $V(G)$ with $[S]_D = V(G)$. Let x be any vertex in a connected graph G . For a vertex y in G , denote by $I_G[y]^x$, the set of all vertices distinct from x that lie on some $x - y$ detour of G ; while for $S \subseteq V(G)$, $I_D[S]^x = \bigcup_{y \in S} I_D[y]^x$. For $x \notin S$, S is an x -detour set of G if $I_D[S]^x = V(G) - \{x\}$ and an x -detour set of minimum cardinality is the x -detour number $d_x(G)$ of G . For $x \notin S$, S is an x -detour convex set if $I_D[S]^x = S$. The x -detour convex hull of S , $[S]_D^x$ is the smallest x -detour convex set containing S . The x -detour hull number $dh_x(G)$ is the minimum cardinality among the subsets S of $V(G) - \{x\}$ with $[S]_D^x = V(G) - \{x\}$. In this paper, we investigate how the detour hull number and the vertex detour hull number of a connected graph are affected by adding a pendant edge.

Key words and Phrases: Detour, detour number, detour hull number, x -detour number, x -detour hull number.

Abstrak. Misalkan x dan y berada di graf terhubung G , jarak detour $D(x, y)$ adalah panjang dari lintasan $x - y$ yang terpanjang di G . Lintasan $x - y$ dengan panjang $D(x, y)$ adalah suatu detour $x - y$. Interval detour tertutup $I_D[x, y]$ memuat x, y dan semua titik yang berada dalam suatu detour $x - y$ dari G ; sedangkan untuk $S \subseteq V(G)$, $I_D[S] = \bigcup_{x, y \in S} I_D[x, y]$. Himpunan titik S adalah suatu himpunan konveks detour jika $I_D[S] = S$. Konveks hull detour $[S]_D$ adalah himpunan konveks detour terkecil yang memuat S . Bilangan hull detour $dh(G)$ adalah kardinalitas minimum diantara sub-subhimpunan S dari $V(G)$ dengan $[S]_D = V(G)$. Misalkan x adalah suatu titik di graf terhubung G . Untuk suatu titik y di G , dinotasikan dengan $I_G[y]^x$, himpunan dari semua titik berbeda dari x yang terletak pada suatu detour $x - y$ dari G ; sedangkan untuk $S \subseteq V(G)$, $I_D[S]^x = \bigcup_{y \in S} I_D[y]^x$. Untuk $x \notin S$, S adalah suatu himpunan detour- x dari G jika $I_D[S]^x = V(G) - \{x\}$ dan suatu himpunan detour- x dengan kardinalitas minimum adalah bilangan detour- x $d_x(G)$ dari G . Untuk $x \notin S$, S adalah suatu himpunan detour- x konveks jika $I_D[S]^x = S$. Konveks hull detour- x dari S , $[S]_D^x$ adalah himpunan konveks detour- x yang memuat S . Bilangan hull detour- x $dh_x(G)$ adalah kardinalitas minimum diantara sub-subhimpunan S dari $V(G) - \{x\}$ dengan $[S]_D^x = V(G) - \{x\}$. Pada paper ini, kami memeriksa pengaruh penambahan sisi anting dari suatu graf terhubung terhadap bilangan hull detour dan bilangan hull detour titik.

Kata kunci: Detour, bilangan detour, bilangan hull detour, bilangan detour- x , bilangan hull detour- x .

1. INTRODUCTION

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies, we refer to [1, 6]. For vertices x and y in a nontrivial connected graph G , the *detour distance* $D(x, y)$ is the length of a longest $x - y$ path in G . An $x - y$ path of length $D(x, y)$ is an $x - y$ *detour*. It is known that the detour distance is a metric on the vertex set $V(G)$. The *detour eccentricity* of a vertex u is $e_D(u) = \max\{D(u, v) : v \in V(G)\}$. The *detour radius*, $rad_D(G)$ of G is the minimum detour eccentricity among the vertices of G , while the *detour diameter*, $diam_D(G)$ of G is the maximum detour eccentricity among the vertices of G . The detour distance and the detour center of a graph were studied in [2]. The *closed detour interval* $I_D[x, y]$ consists of x, y , and all vertices lying on some $x - y$ detour of G ; while for $S \subseteq V(G)$, $I_D[S] = \bigcup_{x, y \in S} I_D[x, y]$; S is a *detour set* if $I_D[S] = V(G)$ and a detour set of minimum cardinality is the *detour number* $dn(G)$ of G . Any detour set of cardinality $dn(G)$ is the *minimum detour set* or *dn-set* of G . A vertex x in G is a *detour extreme vertex* if it is an initial or terminal vertex of any detour containing x . The detour number of a graph was introduced in [3] and further studied in [4, 8]. These concepts have interesting applications in Channel Assignment Problem in radio technologies [5, 7].

A set S of vertices of a graph G is a *detour convex set* if $I_D[S] = S$. The *detour convex hull* $[S]_D$ of S is the smallest detour convex set containing S . The detour convex hull of S can also be formed from the sequence $\{I_D^k[S], k \geq 0\}$, where $I_D^0[S] = S, I_D^1[S] = I_D[S]$ and $I_D^k = I_D[I_D^{k-1}[S]]$. From some term on, this sequence must be constant. Let p be the smallest number such that $I_D^p[S] = I_D^{p+1}[S]$. Then $I_D^p[S]$ is the *detour convex hull* $[S]_D$ and we call p as the *detour iteration number* $dn(S)$ of S . A set S of vertices of G is a *detour hull set* if $[S]_D = V(G)$ and a detour hull set of minimum cardinality is the *detour hull number* $dh(G)$. The detour hull number of a graph was introduced and studied in [11].

For the graph G given in Figure 1, and $S = \{v_1, v_6\}$, $I_D[S] = V - \{v_7\}$ and $I_D^2[S] = V$. Thus S is a minimum detour hull set of G and so $dh(G) = 2$. Since S is not a detour set and $S \cup \{v_7\}$ is a detour set of G , it follows from Theorem 1.2 that $dn(G) = 3$. Hence the detour number and detour hull number of a graph are different. Note that the sets $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_2, v_3, v_4, v_5, v_7\}$ are detour convex sets in G . Let x be any vertex of G . For a vertex y in G , $I_G[y]^x$ denotes

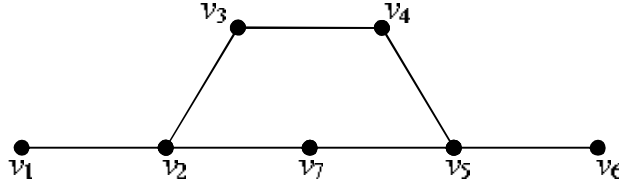
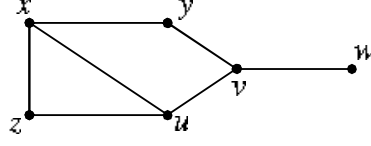


FIGURE 1. Graph G with $dh(G) = 2$ and $dn(G) = 3$

the set of all vertices distinct from x that lie on some $x - y$ detour of G ; while for $S \subseteq V(G)$, $I_D[S]^x = \bigcup_{y \in S} I_D[y]^x$. It is clear that $I_D[x]^x = \phi$. For $x \notin S$, S is an *x -detour set* if $I_D[S]^x = V(G) - \{x\}$ and an *x -detour set* of minimum cardinality is the *x -detour number* $d_x(G)$ of G . Any *x -detour set* of cardinality $d_x(G)$ is the *minimum x -detour set* or *d_x -set* of G . The vertex detour number of a graph was introduced and studied in [9].

Let G be a connected graph and x a vertex in G . Let S be a set of vertices in G such that $x \notin S$. Then S is an *x -detour convex set* if $I_D[S]^x = S$. The *x -detour convex hull* of S , $[S]_D^x$ is the smallest *x -detour convex set* containing S . The *x -detour convex set* can also be formed from the sequence $\{I_D^k[S]^x, k \geq 0\}$, where $I_D^0[S]^x = S, I_D^1[S]^x = I_D[S]^x$ and $I_D^k[S]^x = I_D[I_D^{k-1}[S]^x]^x$. From some term on, this sequence must be constant. Let p_x be the smallest number such that $I_D^{p_x}[S]^x = I_D^{p_x+1}[S]^x$. Then $I_D^{p_x}[S]^x$ is the *x -detour convex hull* $[S]_D^x$ of S and we call p_x as the *x -detour iteration number* $dn_x(S)$ of S . The set S is an *x -detour hull set* if $[S]_D^x = V(G) - \{x\}$ and an *x -detour hull set* of minimum cardinality is the *x -detour hull number* $dh_x(G)$ of G . Any *x -detour hull set* of cardinality $dh_x(G)$ is the *minimum x -detour hull set* or *d_x -hull set* of G .

For the graph G in Figure 2, the minimum vertex detour hull numbers and vertex detour numbers are given in Table 1. Table 1 shows that, for a vertex x , the *x -detour number* and the *x -detour hull number* of a graph are different.

FIGURE 2. G Table 1. x -detour numbers and x -detour hull numbers of G in Figure 2

Vertex	Minimum vertex detour sets	Minimum vertex detour hull sets	Vertex detour number	Vertex detour hull number
x	$\{y, w\}, \{z, w\}, \{u, w\}$	$\{w\}$	2	1
y	$\{w\}$	$\{w\}$	1	1
z	$\{w\}$	$\{w\}$	1	1
u	$\{w\}$	$\{w\}$	1	1
v	$\{y, w\}, \{z, w\}, \{u, w\}$	$\{x, w\}, \{y, w\}, \{z, w\}, \{u, w\}$	2	2
w	$\{y\}, \{z\}, \{u\}$	$\{x\}, \{y\}, \{z\}, \{u\}$	1	1

It is clear that every minimum x -detour hull set of a connected graph G of order n contains at least one vertex and at most $n - 1$ vertices. Also, since every x -detour set is a x -detour hull set, we have the following proposition. Throughout this paper G denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

Theorem 1.1. [11] *Let G be a connected graph. Then*

- (i) *Each detour extreme vertex of G belongs to every detour hull set of G .*
- (ii) *No cut vertex of G belongs to any minimum detour hull set of G .*

Theorem 1.2. [9] *Each end vertex of G other than x (whether x is an end vertex or not) belongs to every minimum x -detour set of G .*

Theorem 1.3. [10] *Let x be a vertex of a connected graph G . Let S be any x -detour hull set of G . Then*

- (i) *Each x -detour extreme vertex of G belongs to S .*
- (ii) *If v is a cut vertex of G and C a component of $G - v$ such that $x \notin V(C)$, then $S \cap V(C) \neq \emptyset$.*
- (iii) *No cut-vertex of G belongs to any minimum x -detour hull set of G .*

Theorem 1.4. [10] *For any vertex x in a connected graph G of order n , $dh_x(G) \leq n - e_D(x)$.*

2. GRAPHS OF ORDER n WITH VERTEX DETOUR HULL NUMBER $n - 1$, $n - 2$ AND $n - 3$

Theorem 2.1. *Let G be a connected graph of order $n \geq 2$. Then $dh_x(G) = n - 1$ for every vertex x in G if and only if $G = K_2$.*

Proof. Suppose that $G = K_2$. Then $dh_x(G) = 1 = n - 1$. The converse follows from Theorem 1.4. \square

Theorem 2.2. *Let G be a connected graph of order $n \geq 3$. Then $dh_x(G) = n - 2$ for every vertex x in G if and only if $G = K_3$.*

Proof. Suppose that $G = K_3$. Then it is clear that $dh_x(G) = 1 = n - 2$ for every vertex x in G . Conversely, suppose that $dh_x(G) = n - 2$ for every vertex x in G . Then by Theorem 1.4, $e_D(x) \leq 2$ for every vertex x in G . It follows from Theorem 2.1 that $e_D(x) \neq 1$ for every vertex x in G . Thus $e_D(x) = 2$ for every vertex x in G ; or the vertex set can be partitioned into V_1 and V_2 such that $e_D(x) = 1$ for $x \in V_1$ and $e_D(x) = 2$ for $x \in V_2$. Thus either $rad_D(G) = diam_D(G) = 2$; or we have $rad_D(G) = 1$ and $diam_D(G) = 2$. This implies that either $G = K_3$ or $G = K_{1,n-1}$. If $G = K_{1,n-1}$, then by Theorem 1.3, $dh_x(G) = n - 1$ for the cut vertex x and $dh_y(G) = n - 2$ for any end vertex y in G , which is a contradiction to the hypothesis. Hence $G = K_3$. \square

Theorem 2.3. *Let G be a connected graph of order $n \geq 2$. Then $G = K_{1,n-1}$ if and only if the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that $dh_x(G) = n - 1$ for $x \in V_1$ and $dh_y(G) = n - 2$ for $y \in V_2$.*

Proof. Suppose that $G = K_{1,n-1}$. Then $dh_x(G) = n - 1$ for the cut vertex x in G and $dh_y(G) = n - 2$ for any end vertex y in G . Conversely, suppose that the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that $dh_x(G) = n - 1$ for $x \in V_1$; and we have $dh_y(G) = n - 2$ for $y \in V_2$. Then by Theorem 1.4, $e_D(x) = 1$ for each $x \in V_1$ and $e_D(y) = 1$ or $e_D(y) = 2$ for each $y \in V_2$. It follows from Theorem 2.1 that $e_D(y) = 2$ for some $y \in V_2$. Hence $rad_D(G) = 1$ and $diam_D(G) = 2$. Thus $G = K_{1,n-1}$. \square

Theorem 2.4. *Let G be a connected graph of order $n \geq 5$. Then G is a double star or $G = K_{1,n-1} + e$ if and only if the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that $dh_x(G) = n - 2$ for $x \in V_1$ and $dh_y(G) = n - 3$ for $y \in V_2$.*

Proof. Suppose that G is a double star or $G = K_{1,n-1} + e$. Then it follows from Theorem 1.3 that $dh_x(G) = n - 2$ or $dh_x(G) = n - 3$ according to whether x is a cut vertex of G or not. Conversely, suppose that $dh_x(G) = n - 2$ for $x \in V_1$ and $dh_x(G) = n - 3$ for $x \in V_2$. Then by Theorem 1.4, $e_D(x) \leq 3$ for every x and so $diam_D(G) \leq 3$. It follows from Theorem 2.1 that $G \neq K_2$ and so $diam_D(G) \geq 2$. If $diam_D(G) = 2$, then G is the star $K_{1,n-1}$ and by Theorem 2.3, $dh_x(G) = n - 1$ or $dh_x(G) = n - 2$ for every vertex x . This is a contradiction to the hypothesis. Now, suppose that $diam_D(G) = 3$. If G is a tree, then G is a double star. If G is not a tree, then it is clear that $3 \leq cir(G) \leq 4$, where $cir(G)$ denotes the length of a longest cycle in G . We prove that $cir(G) = 3$. Suppose that $cir(G) = 4$. Let $C_4 : v_1, v_2, v_3, v_4, v_1$ be a 4-cycle in G . Since $n \geq 5$ and G is connected, there is a vertex x not on C_4 such that x is adjacent to some vertex say, v_1 of G . Then x, v_1, v_2, v_4, v_4 is a path of length 4 in G and so $diam_D(G) \geq 4$, which

is a contradiction. Thus $\text{cir}(G) = 3$. Also, if G contains two or more cycles, then it follows that $\text{diam}_D(G) \geq 4$. Hence G contains a unique triangle, say $C_3 : v_1, v_2, v_3, v_1$. Since $n \geq 5$, at least one vertex of C_3 has degree at least 3. If there are two or more vertices of C_3 having degree at least 3, then $\text{diam}_D(G) \geq 4$, which is a contradiction. Thus exactly one vertex of C_3 has degree at least 3 and it follows that $G = K_{1,n-1} + e$. This completes the proof. \square

3. DETOUR AND VERTEX DETOUR HULL NUMBERS AND ADDITION OF A PENDANT EDGE

In this section we discuss how the detour hull number and the vertex detour hull number of a connected graph are affected by adding a pendant edge to G . Let G' be a graph obtained from a connected graph G by adding a pendant edge uv , where u is not a vertex of G and v is a vertex of G .

Theorem 3.1. *If G' is a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G , then $d_h(G) \leq d_h(G') \leq d_h(G) + 1$.*

Proof. Let S be a minimum detour hull set of G and let $S' = S \cup \{u\}$. We show that S' is a detour hull set of G' . Let $x \in V(G')$. If $x = u$, then $x \in S'$. So, assume that $x \in V(G)$. Then $x \in I_D^k[S]_G$ for some $k \geq 0$. Since $I_D^n[S]_G = I_D^n[S]_{G'}$ for all $n \geq 0$, we have $x \in I_D^k[S]_{G'}$. Also, since $S \subseteq S'$, we see that $I_D^n[S]_{G'} \subseteq I_D^n[S']_{G'}$ for all $n \geq 0$. Hence $x \in I_D^k[S']_{G'}$. This implies that S' is a detour hull set of G' so that $d_h(G') \leq |S'| = |S| + 1 = d_h(G) + 1$. For the lower bound, let S' be a minimum detour hull set of G' . Then by Theorem 1.1, $u \in S'$ and $v \notin S'$. Let $S = (S' - \{u\}) \cup \{v\}$. We prove that S is a detour hull set of G . For this, first we claim that $I_D^k[S']_{G'} - \{u\} \subseteq I_D^k[S]_G$ for all $k \geq 0$. We use induction on k . Since $S' - \{u\} \subseteq S$, the result is true for $k = 0$. Let $k = 1$ and let $x \in I_D[S']_{G'} - \{u\}$. Then $x \neq u$. If $x \in S'$, then $x \in S \subseteq I_D[S]_G$. If $x \notin S'$, then there exist $y, z \in S'$ such that $x \in I_D[y, z]_{G'}$ with $x \neq y, z$. If $y \neq u$ and $z \neq u$, then $y, z \in S$ and so $I_D[y, z]_G = I_D[y, z]_{G'}$. Thus $x \in I_D[S]_G$. Now, let $y = u$ or $z = u$, say $z = u$. Since v is a cut vertex of G' , it follows that $x \in I_D[y, v]_{G'} = I_D[y, v]_G$ and hence $x \in I_D[S]_G$. Assume that the result is true for $k = l$. Then $I_D^l[S']_{G'} - \{u\} \subseteq I_D^l[S]_G$. Now, let $x \in I_D^{l+1}[S']_{G'} - \{u\}$. If $x \in I_D^l[S']_{G'}$, then by induction hypothesis, we have $x \in I_D^l[S]_G \subseteq I_D^{l+1}[S]_G$. If $x \notin I_D^l[S']_{G'}$, then there exist $y, z \in I_D^l[S']_{G'}$ such that $x \in I_D[y, z]_{G'}$ with $x \neq y, z$. If $y \neq u$ and $z \neq u$, then it follows from induction hypothesis that $y, z \in I_D^l[S]_G$. Also, since $I_D[y, z]_{G'} = I_D[y, z]_G$, we have $x \in I_D^{l+1}[S]_G$. Let $y = u$ or $z = u$, say $z = u$. Then $y \neq u$ and so by induction hypothesis, $y \in I_D^l[S]_G$. Since v is a cut vertex of G' , it follows that $x \in I_D[y, v]_{G'} = I_D[y, v]_G$. Also, since $v \in S \subseteq I_D^l[S]_G$, it follows that $x \in I_D^{l+1}[S]_G$. Hence the proof of the claim is complete by induction. Now, since S' is a minimum detour hull set of G' , there is an integer $r \geq 0$ such that $I_D^r[S']_{G'} = V(G')$ and it follows from the above claim that $I_D^r[S]_G = V(G)$. Thus S is a detour hull set of G so that $d_h(G) \leq |S| = |S'| = d_h(G')$. This completes the proof. \square

Remark 3.2. The bounds for $d_h(G')$ in Theorem 3.1 are sharp. Let G' be the graph obtained from the graph G in Figure 3, by adding a pendant edge at one of its end vertices. Then $d_h(G') = d_h(G) = 2$. If G' is obtained from G by adding a pendant edge at one of its cut vertices, then $d_h(G') = d_h(G) + 1$.

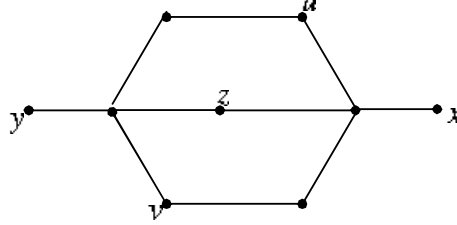


FIGURE 3. Graph G with $d_h(G') = d_h(G) + 1$

Theorem 3.3. Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G . Then $d_h(G) = d_h(G')$ if and only if v is a vertex of some minimum detour hull set of G .

Proof. First, assume that there is a minimum detour hull set S of G such that $v \in S$. Let $S' = (S - \{v\}) \cup \{u\}$. Then $|S'| = |S|$. We show that S' is a detour hull set of G' . First, we claim that $I_D^k[S]_G \subseteq I_D^{k+1}[S']_{G'}$ for all $k \geq 0$. We prove this by using induction on k . Let $k = 0$. Let $x \in S$. If $x \neq v$, then $x \in S' \subseteq I_D[S']_{G'}$. If $x = v$, then $x \in I_D[y, u]_{G'} \subseteq I_D[S']_{G'}$, where $y \in S$ such that $y \neq v$. Thus $S \subseteq I_D[S']_{G'}$. Assume the result for $k = l$. Then $I_D^l[S]_G \subseteq I_D^{l+1}[S']_{G'}$. Let $x \in I_D^{l+1}[S]_G$. If $x \in I_D^l[S]_G$, then by induction hypothesis, $x \in I_D^{l+1}[S']_{G'} \subseteq I_D^{l+2}[S']_{G'}$. If $x \notin I_D^l[S]_G$, then there exist $y, z \in I_D^l[S]_G$ such that $x \in I_D[y, z]_G = I_D[y, z]_{G'}$. By induction hypothesis, we have $y, z \in I_D^{l+1}[S']_{G'}$ and so $x \in I_D^{l+2}[S']_{G'}$. Hence by induction $I_D^k[S]_G \subseteq I_D^{k+1}[S']_{G'}$ for all $k \geq 0$. Now, since S is a detour hull set of G , there exists an integer $r \geq 0$ such that $I_D^r[S]_G = V(G)$ and it follows from the above claim that $I_D^{r+1}[S']_{G'} = V(G')$. Thus S' is a detour hull set of G so that $d_h(G') \leq |S'| = |S| = d_h(G)$. The other inequality follows from Theorem 3.1.

Conversely, let $d_h(G) = d_h(G')$. Let S' be a minimum detour hull set of G' . Then by Theorem 1.3, $u \in S'$ and $v \notin S'$. Let $S = (S' - \{u\}) \cup \{v\}$. Then, as in the proof of Theorem 3.1, we can prove that S is a detour hull set of G . Since $|S| = |S'| = d_h(G') = d_h(G)$, we see that S is a minimum detour hull set of G and $v \in S$. This completes the proof. \square

Theorem 3.4. Let G be a connected graph and let x be any vertex in G . If G' is a graph obtained from G by adding a pendant edge xu , then $dh_x(G') = dh_x(G) + 1$.

Proof. Let S be a minimum x -detour hull set of G and let $S' = S \cup \{u\}$. Then, as in Theorem 3.1, it is straight forward to verify that $I_D^n[S]_G^x \subseteq I_D^n[S']_{G'}^x$ for all $n \geq 0$. Since S is an x -detour hull set of G , there is an integer $r \geq 0$ such that

$I_D^r[S]_G^x = V(G) - \{x\}$ and it is clear that $I_D^r[S']_{G'}^x = V(G') - \{x\}$. Hence S' is an x -detour hull set of G' so that $dh_x(G') \leq |S'| = dh_x(G) + 1$. Now, suppose that $dh_x(G') < dh_x(G) + 1$. Let S' be a minimum x -detour hull set of G' . Then, by Theorem 1.3, $u \in S'$. Let $S = S' - \{u\}$. Then, as in Theorem 3.1, it is straight forward to prove that $I_D^n[S']_{G'}^x - \{u\} \subseteq I_D^n[S]_G^x$ for all $n \geq 0$. Since S' is an x -detour hull set of G' , there is an integer $r \geq 0$ such that $I_D^r[S']_{G'}^x = V(G') - \{x\}$. Hence $I_D^r[S]_G^x = V(G) - \{x\}$. Thus S is an x -detour hull set of G so that $dh_x(G) \leq |S| = dh_x(G') - 1$, which is a contradiction to $dh_x(G') < dh_x(G) + 1$. Hence the result follows. \square

Theorem 3.5. *Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G . Then $dh_u(G') = dh_v(G)$.*

Proof. Let S be a minimum v -detour hull set of G . Then $v \notin S$. As in Theorem 3.1, it is straight forward to prove that $I_D^n[S]_G^v \subseteq I_D^n[S]_{G'}^u$ for all $n \geq 0$. Since S is a v -detour hull set of G , there is an integer $r \geq 0$ such that $I_D^r[S]_G^v = V(G) - \{v\}$. Now, since $v \in I_D[z]_G^u$ for any $z \in S$, it follows that $I_D^r[S]_{G'}^u = V(G') - \{u\}$. Hence S is a u -detour hull set of G' so that $dh_u(G') \leq |S| = dh_v(G)$. For the other inequality, let T be a minimum u -detour hull set of G' . Then $u \notin T$ and by Theorem 1.3(iii), $v \notin T$. As in Theorem 3.1, it is straight forward to prove that $I_D^n[T]_{G'}^u - \{v\} \subseteq I_D^n[T]_G^v$ for all $n \geq 0$. Since T is a u -detour hull set of G' , there is an integer $r \geq 0$ such that $I_D^r[T]_{G'}^u = V(G') - \{u\}$. Hence it follows that $I_D^r[T]_G^v = V(G) - \{v\}$ and T is a v -detour hull set of G . Thus $dh_v(G) \leq |T| = dh_u(G')$. This completes the proof. \square

Theorem 3.6. *Let G be a connected graph and x any vertex of G . Let G' be a graph obtained from G by adding a pendant edge uv at a vertex $v \neq x$ of G . Then $dh_x(G) \leq dh_x(G') \leq dh_x(G) + 1$.*

Proof. The proof is similar to Theorem 3.1. \square

Theorem 3.7. *Let G be a connected graph and x any vertex of G . Let G' be a graph obtained from G by adding a pendant edge uv at a vertex $v \neq x$ of G . Then $dh_x(G) = dh_x(G')$ if and only if v belongs to some minimum x -detour hull set of G .*

Proof. The proof is similar to Theorem 3.3. \square

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